

Approximating the inverse of a symmetric matrix with non-negative elements

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Abstract

For an $n \times n$ symmetric positive definite matrix $T = (t_{i,j})$ with positive elements satisfying $t_{i,i} \geq \sum_{j \neq i} t_{i,j}$ and certain bounding conditions, we propose to use the matrix $S = (s_{i,j})$ to approximate its inverse, where $s_{i,j} = \delta_{i,j}/t_{i,i} - 1/t_{..}$, $\delta_{i,j}$ is the Kronecker delta function, and $t_{..} = \sum_{i,j=1}^n (1 - \delta_{i,j})t_{i,j}$. An explicit bound on the approximation error is obtained, showing that the inverse is well approximated to order $1/(n-1)^2$ uniformly. The results are further extended to allow some off-diagonal elements of T to be zeros.

Keywords: Approximation error; Inverse; Symmetric; Non-negative elements.

1. Introduction

When solving the solution for a large system of linear equations, a good approximate inverse of the coefficient matrix is crucially important in establishing fast convergence rates for iterative algorithms. See the extensive reviews: [1, 3, 5, 18]. Here, we are concerned with $n \times n$ symmetric diagonally dominant matrices $T = (t_{i,j})$ with positive elements, i.e.,

$$t_{i,j} = t_{j,i} > 0 \quad \text{and} \quad t_{i,i} \geq \sum_{j=1, j \neq i}^n t_{i,j}, \quad i = 1, \dots, n. \quad (1)$$

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This kind of diagonally dominant nonnegative matrices has received wide attention [6, 8, 10]. It is easy to show that T must be positive definite. The problems on inverses of nonnegative matrices have been extensively discussed [2, 9, 7]. Markham [11] and Martinez, Michon and San Martin [12] studied the sufficient conditions that the inverse of a nonnegative matrix is an M -matrix.

In this paper, we propose to approximate the inverse of T , T^{-1} , by the matrix $S = (s_{i,j})$, where

$$s_{i,j} = \frac{\delta_{i,j}}{t_{i,i}} - \frac{1}{t_{..}},$$

where $t_{..} = \sum_{i,j=1}^n (1 - \delta_{i,j}) t_{i,j}$. An explicit upper bound on the approximation error is given in the following section, which is crucially used to establish the asymptotical normality of an estimated vector and a Wilks type of theorem in the β -model for undirected random graphs with a diverging number of nodes [16, 17].

2. An explicit bound on the approximation error

Let $m := \min_{1 \leq i < j \leq n} t_{i,j}$, $\Delta_i := t_{i,i} - \sum_{j \neq i} t_{i,j}$, $M := \max\{\max_{1 \leq i < j \leq n} t_{i,j}, \max_{1 \leq i \leq n} \Delta_i\}$, and for a matrix $A = (a_{i,j})$, define $\|A\| := \max_{i,j} |a_{i,j}|$. We have the following theorem.

Theorem 1.

$$\|T^{-1} - S\| \leq \frac{C(m, M)}{(n-1)^2},$$

where

$$C(m, M) = \left(\frac{n(M/m) + (n-2)}{2(n-2)} - \frac{(n-2)M/m}{((n-2) + M/m)((n-2) + 2M/m)} \right) \left(\frac{M}{m^2} + \frac{3M}{m^2 n} \right) + \frac{1}{2m}.$$

Proof. Let I_n be the $n \times n$ identity matrix. Define $F = T^{-1} - S$, $V = (v_{ij}) = I_n - TS$ and $W = (w_{ij}) = SV$. We have the recursion

$$F = T^{-1} - S = (T^{-1} - S)(I_n - TS) + S(I_n - TS) = FV + W. \quad (2)$$

Note that

$$\begin{aligned}
v_{i,j} &= \delta_{i,j} - \sum_{k=1}^n t_{i,k} s_{k,j} \\
&= \delta_{i,j} - \sum_{k=1}^n t_{i,k} \left(\frac{\delta_{k,j}}{t_{j,j}} - \frac{1}{t_{..}} \right) \\
&= (\delta_{i,j} - 1) \frac{t_{i,j}}{t_{j,j}} + \frac{2t_{i,i} - \Delta_i}{t_{..}}, \tag{3}
\end{aligned}$$

and

$$\begin{aligned}
w_{i,j} &= \sum_{k=1}^n s_{i,k} v_{k,j} = \sum_{k=1}^n \left(\frac{\delta_{i,k}}{t_{i,i}} - \frac{1}{t_{..}} \right) \left[(\delta_{k,j} - 1) \frac{t_{k,j}}{t_{j,j}} + \frac{2t_{k,k} - \Delta_k}{t_{..}} \right] \\
&= \sum_{k=1}^n \frac{\delta_{i,k}}{t_{i,i}} \left[(\delta_{k,j} - 1) \frac{t_{k,j}}{t_{j,j}} + \frac{2t_{k,k} - \Delta_k}{t_{..}} \right] - \frac{1}{t_{..}} \sum_{k=1}^n \left[(\delta_{k,j} - 1) \frac{t_{k,j}}{t_{j,j}} + \frac{2t_{k,k} - \Delta_k}{t_{..}} \right] \\
&= \left[\frac{(\delta_{i,j} - 1)}{t_{i,i}} \left(\frac{t_{i,j}}{t_{j,j}} \right) + \frac{2t_{i,i} - \Delta_i}{t_{i,i}t_{..}} \right] - \frac{1}{t_{..}} \left[\frac{-(t_{j,j} - \Delta_j)}{t_{j,j}} + 2 + \frac{\sum_k \Delta_k}{t_{..}} \right] \\
&= \frac{(\delta_{i,j} - 1)t_{i,j}}{t_{i,i}t_{j,j}} + \frac{1}{t_{..}} - \frac{\Delta_i}{t_{i,i}t_{..}} - \frac{\Delta_j}{t_{j,j}t_{..}} - \frac{\sum_k \Delta_k}{t_{..}^2}. \tag{4}
\end{aligned}$$

Furthermore, when $i \neq j$,

$$\begin{aligned}
0 &< \frac{1}{t_{..}} \leq \frac{1}{mn(n-1)}, \\
0 &< \frac{t_{i,j}}{t_{i,i}t_{j,j}} \leq \frac{M}{m^2(n-1)^2}, \\
0 &< \frac{\Delta_i}{t_{i,i}t_{..}} \leq \frac{M}{m^2n(n-1)^2},
\end{aligned}$$

and it is easy to show, when i, j, k are different from each other,

$$\begin{aligned}
|w_{i,i}| &\leq \max\left\{ \frac{1}{mn(n-1)}, \frac{3M}{m^2n(n-1)^2} \right\}, \\
|w_{i,j}| &\leq \max\left\{ \frac{1}{mn(n-1)}, \frac{M}{m^2(n-1)^2} + \frac{3M}{m^2n(n-1)^2} \right\}, \\
|w_{i,j} - w_{i,k}| &\leq \frac{M}{m^2(n-1)^2} + \frac{M}{m^2n(n-1)^2}, \\
|w_{i,i} - w_{i,k}| &\leq \frac{M}{m^2(n-1)^2} + \frac{M}{m^2n(n-1)^2}.
\end{aligned}$$

It follows that

$$\max(|w_{i,j}|, |w_{i,j} - w_{i,k}|) \leq \frac{M}{m^2(n-1)^2} + \frac{3M}{m^2n(n-1)^2} \quad \text{for all } i, j, k. \quad (5)$$

Next we use the recursion (2) to obtain a bound of the approximate error $\|F\|$. Let $a = \frac{M}{m^2(n-1)^2} + \frac{3M}{m^2n(n-1)^2}$. By (2) and (3), for any i , we have

$$f_{i,j} = \sum_{k=1}^n f_{i,k}[(\delta_{k,j} - 1)\frac{t_{k,j}}{t_{j,j}} + \frac{2t_{k,k} - \Delta_k}{t_{..}}] + w_{i,j}, \quad j = 1, \dots, n. \quad (6)$$

Thus, to prove Theorem 1, it is sufficient to show that $|f_{i,j}| \leq C(M, m)/(n-1)^2$ for any i, j . Fixing any i , let $f_{i,\alpha} = \max_{1 \leq k \leq n} f_{i,k}$ and $f_{i,\beta} = \min_{1 \leq k \leq n} f_{i,k}$.

Without loss of generality, we assume that $f_{i,\alpha} \geq |f_{i,\beta}|$.

First, we will show that $f_{i,\beta} \leq 1/t_{..} \leq 1/(m(n-1)^2)$. A direct calculation gives that

$$\begin{aligned} \sum_{k=1}^n f_{i,k} t_{k,i} &= \sum_{k=1}^n (T_{i,k}^{-1} - (\frac{\delta_{i,k}}{t_{i,i}} - \frac{1}{t_{..}})) t_{k,i} \\ &= 1 - (1 - \sum_{k=1}^n \frac{t_{k,i}}{t_{..}}) = \sum_{k=1}^n \frac{t_{k,i}}{t_{..}}. \end{aligned} \quad (7)$$

Thus, $f_{i,\beta} \sum_{k=1}^n t_{k,i} \leq \sum_{k=1}^n f_{i,k} t_{k,i} = \sum_{k=1}^n \frac{t_{k,i}}{t_{..}}$. It follows that $f_{i,\beta} \leq 1/t_{..}$ and, similarly, $f_{i,\alpha} \geq 1/t_{..}$.

Note that $(1 - \Delta_\alpha/t_{\alpha,\alpha})f_{i,\beta} = -\sum_{k=1}^n f_{i,\beta}(\delta_{k,\alpha} - 1)\frac{t_{k,\alpha}}{t_{\alpha,\alpha}}$. Thus,

$$f_{i,\alpha} + (1 - \frac{\Delta_\alpha}{t_{\alpha,\alpha}})f_{i,\beta} = \sum_{k=1}^n (f_{i,k} - f_{i,\beta})(\delta_{k,\alpha} - 1)\frac{t_{k,\alpha}}{t_{\alpha,\alpha}} + \sum_{k=1}^n f_{i,k}[\frac{2t_{k,k} - \Delta_k}{t_{..}}] + w_{i,\alpha}. \quad (8)$$

Similarly, we have that

$$f_{i,\beta} + (1 - \frac{\Delta_\alpha}{t_{\alpha,\alpha}})f_{i,\beta} = \sum_{k=1}^n (f_{i,k} - f_{i,\beta})(\delta_{k,\beta} - 1)\frac{t_{k,\beta}}{t_{\beta,\beta}} + \sum_{k=1}^n f_{i,k}[\frac{2t_{k,k} - \Delta_k}{t_{..}}] + w_{i,\beta}. \quad (9)$$

Combining the above two equations, it yields

$$f_{i,\alpha} - f_{i,\beta} = \sum_{k=1}^n (f_{i,k} - f_{i,\beta})[(\delta_{k,\alpha} - 1)\frac{t_{k,\alpha}}{t_{\alpha,\alpha}} - (\delta_{k,\beta} - 1)\frac{t_{k,\beta}}{t_{\beta,\beta}}] + w_{i,\alpha} - w_{i,\beta}. \quad (10)$$

Let $\Omega = \{k : (1 - \delta_{k,\beta})t_{k,\beta}/t_{\beta,\beta} \geq (1 - \delta_{k,\alpha})t_{k,\alpha}/t_{\alpha,\alpha}\}$ and let $|\Omega| = \lambda$. Note that $1 \leq \lambda \leq n - 1$. Then,

$$\begin{aligned}
& \sum (f_{i,k} - f_{i,\beta})[(\delta_{k,\alpha} - 1)\frac{t_{k,\alpha}}{t_{\alpha,\alpha}} - (\delta_{k,\beta} - 1)\frac{t_{k,\beta}}{t_{\beta,\beta}}] \\
& \leq \sum_{k \in \Omega} (f_{i,k} - f_{i,\beta})[(1 - \delta_{k,\beta})\frac{t_{k,\beta}}{t_{\beta,\beta}} - (1 - \delta_{k,\alpha})\frac{t_{k,\alpha}}{t_{\alpha,\alpha}}] \\
& \leq (f_{i,\alpha} - f_{i,\beta})[\frac{\sum_{k \in \Omega} t_{k,\beta}}{t_{\beta,\beta}} - \frac{\sum_{k \in \Omega} (1 - \delta_{k,\alpha})t_{k,\alpha}}{t_{\alpha,\alpha}}] \\
& \leq (f_{i,\alpha} - f_{i,\beta})[\frac{\lambda M}{\lambda M + (n - 1 - \lambda)m} - \frac{(\lambda - 1)m}{(\lambda - 1)m + (n - \lambda)M + M}] \quad (11)
\end{aligned}$$

Let

$$f(\lambda) = \frac{\lambda M}{\lambda M + (n - 1 - \lambda)m} - \frac{(\lambda - 1)m}{(\lambda - 1)m + (n - \lambda)M}.$$

There are two cases to consider the maximum of $f(\lambda)$ in the range of $\lambda \in [1, n - 1]$.

Case I: When $M = m$, it is easy to show $f(\lambda) = 1/(n - 1)$.

Case II: $M \neq m$. Since

$$\begin{aligned}
f'(\lambda) &= \frac{(n - 1)Mm}{[\lambda M + (n - 1 - \lambda)m]^2} - \frac{(n - 1)Mm}{[(\lambda - 1)m + (n - \lambda)M]^2} \\
&= \frac{(n - 1)Mm[(n - 2\lambda)(M - m)][\lambda M + (n - 1 - \lambda)m + (\lambda - 1)m + (n - \lambda)M]}{[\lambda M + (n - 1 - \lambda)m]^2[(\lambda - 1)m + (n - \lambda)M]^2}
\end{aligned}$$

and

$$f''(\lambda) = -2(M - m)Mm(n - 1) \left(\frac{1}{[\lambda M + (n - 1 - \lambda)m]^3} + \frac{1}{[(\lambda - 1)m + (n - \lambda)M]^3} \right),$$

$f(\lambda)$ takes its maximum at $\lambda = n/2$ when $1 \leq \lambda \leq n - 1$. A direct calculation gives that

$$f\left(\frac{n}{2}\right) = \frac{nM - (n - 2)m}{nM + (n - 2)m}. \quad (12)$$

Moreover, let

$$g(\lambda) = \frac{(\lambda - 1)m}{(\lambda - 1)m + (n - \lambda)M} - \frac{(\lambda - 1)m}{(\lambda - 1)m + (n - \lambda)M + M}.$$

Since

$$g'(\lambda) = \frac{Mm[M^2((n-\lambda)^2 + 2(n-\lambda)(\lambda-1) + n-1) + (2Mm - m^2)(\lambda-1)^2]}{[(\lambda-1)m + (n-\lambda)M]^2[(\lambda-1)m + (n-\lambda)M + M]^2},$$

$g'(\lambda) > 0$ when $1 \leq \lambda \leq n-1$ such that for $1 \leq \lambda \leq n-1$,

$$0 \leq g(\lambda) \leq \frac{(n-2)Mm}{[(n-2)m + M][(n-2)m + 2M]}. \quad (13)$$

By (12) and (13), we have

$$\begin{aligned} & \max_{1 \leq \lambda \leq n-1} \left[\frac{\lambda M}{\lambda M + (n-1-\lambda)m} - \frac{(\lambda-1)m}{(\lambda-1)m + (n-\lambda)M + M} \right] \\ & \leq \max_{1 \leq \lambda \leq n-1} f(\lambda) + \max_{1 \leq \lambda \leq n-1} g(\lambda) \\ & \leq \frac{2}{n} I(M=m) + \frac{nM - (n-2)m}{nM + (n-2)m} I(M \neq m) + \frac{(n-2)Mm}{[(n-2)m + M][(n-2)m + 2M]} \\ & = \frac{nM - (n-2)m}{nM + (n-2)m} + \frac{(n-2)Mm}{[(n-2)m + M][(n-2)m + 2M]}, \end{aligned} \quad (14)$$

where $I(\cdot)$ is an indicator function. Combining (11) and (14), it yields

$$f_{i,\alpha} - f_{i,\beta} \leq \left(\frac{nM - (n-2)m}{nM + (n-2)m} + \frac{(n-2)Mm}{[(n-2)m + M][(n-2)m + 2M]} \right) \times (f_{i,\alpha} - f_{i,\beta}) + a,$$

so that

$$f_{i,\alpha} - f_{i,\beta} \leq \left[\frac{nM + (n-2)m}{2(n-2)m} - \frac{(n-2)Mm}{[(n-2)m + M][(n-2)m + 2M]} \right] \times a.$$

Consequently,

$$\begin{aligned} f_{i,\alpha} & \leq \left[\frac{nM + (n-2)m}{2(n-2)m} - \frac{(n-2)Mm}{[(n-2)m + M][(n-2)m + 2M]} \right] \\ & \quad \times \left[\frac{M}{m^2(n-1)^2} + \frac{3M}{m^2n(n-1)^2} \right] + \frac{1}{2m(n-1)^2} \\ & = C(M, m)/(n-1)^2. \end{aligned}$$

This completes the proof.

Now we extend the results to allow some off-diagonal elements of T to be zeros. In this case, we redefine $m = \min_{(i,j) \in \{(i,j): t_{i,j} > 0\}} t_{i,j}$. Moreover, define $n_i = \sum_{j=1; j \neq i}^n I(t_{i,j} > 0)$ and $\rho_{\max} = \max_i n_i / (n - 1)$ and $\rho_{\min} = \min_i n_i / (n - 1)$.

Condition A: Assume that

$$T = \begin{pmatrix} T_{k_1, k_1}^{(1)} & * & \cdots & * \\ * & T_{k_2, k_2}^{(2)} & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & T_{k_m, k_m}^{(m)} \end{pmatrix}$$

where $T_{k_i, k_i}^{(i)}$ is a sub-matrix consisting of all positive elements, for every $i = 1, \dots, m$, and $\min_{i=1, \dots, m} (k_i - 2)/n \geq \tau$.

Then under Condition A, we have the following theorem.

Theorem 2. Assume $\sum_{j=1}^n (\delta_{i,j} - 1)t_{i,j} = 0$ for all i . Under Condition A, if $\tau > 0$, then

$$\|T^{-1} - S\| \leq \frac{C(m, M, \rho_{\max}, \rho_{\min}, \tau)}{(n - 1)^2},$$

where

$$C(m, M, \rho_{\max}, \rho_{\min}, \tau) = \left(\frac{4M^2 \rho_{\max} (\rho_{\max} - \tau/2)}{m^2 \tau^2} + \frac{4\rho_{\max} M}{m\tau} \right) \times \left(\frac{M}{\rho_{\min}^2 m^2} + \frac{1}{2\rho_{\min}^2 m} \right).$$

Proof. By the assumption $\sum_{j=1}^n (\delta_{i,j} - 1)t_{i,j} = 0$ for all i , we have that $\Delta_k = 0$ and $2t_{k,k} = \sum_{d=1}^n t_{k,d}$ such that

$$\begin{aligned} 2 \sum_{k=1}^n f_{i,k} t_{k,k} &= \sum_{k=1}^n \sum_{d=1}^n f_{i,k} t_{k,d} \\ &= \sum_{k=1}^n \sum_{d=1}^n T_{i,k}^{-1} t_{k,d} - \sum_{d=1}^n \sum_{k=1}^n \left(\frac{\delta_{i,k}}{t_{i,i}} - \frac{1}{t_{..}} \right) t_{k,d} \\ &= \sum_{d=1}^n \delta_{i,d} - \sum_{d=1}^n \left(\frac{t_{i,d}}{t_{i,i}} - \frac{2t_{d,d}}{t_{..}} \right) = 1. \end{aligned}$$

This shows

$$\sum_{k=1}^n \frac{2f_{i,k}t_{k,k}}{t_{..}} = \frac{1}{t_{..}}. \quad (15)$$

In view of (6) and (15), it yields

$$f_{i,\alpha} = \sum_{k=1}^n f_{i,k}(\delta_{k,\alpha} - 1) \frac{t_{k,\alpha}}{t_{\alpha,\alpha}} + w_{i,\alpha} + \frac{1}{t_{..}}. \quad (16)$$

Similar to the inequality (5), when some entries of T are zero, we have the following bound for $\max\{|w_{i,j}|, |w_{i,j} - w_{i,k}|\}$:

$$\max(|w_{i,j}|, |w_{i,j} - w_{i,k}|) \leq \frac{M}{m^2 \rho_{\min}^2 (n-1)^2} \quad \text{for all } i, j, k. \quad (17)$$

Define $f_{i,\alpha}$ and $f_{i,\beta}$ as in the proof of Theorem 2.1 and fix i . Similarly, the equation (7) holds and $f_{i,\beta} \leq 1/(2t_{..}) \leq 1/(2m\rho_{\min}^2(n-1)^2)$. Without loss of generality, we assume that $f_{i,\alpha} \geq |f_{i,\beta}|$.

Let $b = (M/m^2 + 1/(2m))/(\rho_{\min}^2(n-1)^2)$. Then $|w_{i,j} + 1/(2t_{..})| \leq b$ for all i, j . Assume that $t_{\alpha,\alpha} \in \mathcal{T}_{k_i, k_i}^{(i)}$, where $\mathcal{T}_{k_i, k_i}^{(i)}$ is a set consisting of all the elements of $T_{k_i, k_i}^{(i)}$. If $\sum_{l=1}^{i-1} k_l < \alpha \leq \sum_{l=1}^i k_l$, then define $\Omega = \{\sum_{l=1}^{i-1} k_l + 1, \dots, \sum_{l=1}^i k_l\} \setminus \{\alpha\}$; otherwise, define $\Omega = \{\sum_{l=1}^{i-1} k_l + 1, \dots, \sum_{l=1}^i k_l\}$. Note that for any $j, l \in \Omega$, $t_{j,l} > 0$. Moreover, let $\Omega_1 = \{k : f_{i,k} \geq 0, k \in \Omega\}$ and $\Omega_2 = \{k : f_{i,k} < 0, k \in \Omega\}$. Then $|\Omega_1| + |\Omega_2| = |\Omega| \geq \tau n + 2$. Thus, at least one of $|\Omega_1|$ and $|\Omega_2|$ is no less than $|\Omega|/2$. Thus, we consider two cases separately.

Case I: $|\Omega_1| \geq |\Omega|/2 \geq \tau n/2 + 1$. Let $f_{i,\alpha_1} = \min_{j \in \Omega_1} f_{i,j} \geq 0$. By (16),

$$b \geq \sum_{k=1}^n (f_{i,\alpha} + f_{i,k})(1 - \delta_{k,\alpha}) \frac{t_{k,\alpha}}{t_{\alpha,\alpha}} \geq (f_{i,\alpha} + f_{i,\alpha_1}) \frac{m(|\Omega_1| - 1)}{n_\alpha M}. \quad (18)$$

It follows that

$$f_{i,\alpha} \leq f_{i,\alpha} + f_{i,\alpha_1} \leq \frac{2\rho_{\max} M b}{m\tau}.$$

Case II: $|\Omega_2| \geq |\Omega|/2 \geq \tau n + 1$. Let $f_{i,\alpha_2} = \min_{j \in \Omega_2} f_{i,j} < 0$. Similar to (18), we have

$$b \geq \sum_{k=1}^n (f_{i,\alpha} + f_{i,k})(1 - \delta_{k,\alpha}) \frac{t_{k,\alpha}}{t_{\alpha,\alpha}} \geq (f_{i,\alpha} + f_{i,\alpha_2}) \frac{m(|\Omega_2| - 1)}{n_\alpha M},$$

and

$$f_{i,\alpha} + f_{i,\alpha_2} \leq \frac{2\rho_{\max}Mb}{m\tau}. \quad (19)$$

By (6) and (7), we have

$$\sum_{k=1}^n (f_{i,\alpha_2} + f_{i,k})(1 - \delta_{k,\alpha_2}) \frac{t_{k,\alpha_2}}{t_{\alpha_2,\alpha_2}} = w_{i,\alpha_2} + \frac{1}{2t_{..}} \geq -b,$$

so that

$$\begin{aligned} -b &\leq \sum_{k=1}^n (f_{i,\alpha_2} + f_{i,k})(1 - \delta_{k,\alpha_2}) \frac{t_{k,\alpha_2}}{t_{\alpha_2,\alpha_2}} \\ &\leq \sum_{k \in \Omega_2 \cap \Omega_3} (f_{i,\alpha_2} + f_{i,\alpha_3}) \frac{t_{k,\alpha_2}}{t_{\alpha_2,\alpha_2}} + \sum_{k \in \Omega_2^c \cap \Omega_3} (f_{i,\alpha_2} + f_{i,k}) \frac{t_{k,\alpha_2}}{t_{\alpha_2,\alpha_2}} \\ &\leq (f_{i,\alpha_2} + f_{i,\alpha_3}) \frac{m|\Omega_2 \cap \Omega_3|}{Mn_{\alpha_2}} + (f_{i,\alpha_2} + f_{i,\alpha}) \frac{M|\Omega_2^c \cap \Omega_3|}{mn_{\alpha_2}}, \end{aligned}$$

where $\Omega_3 = \{j : t_{\alpha_2,j} > 0; j \neq \alpha_2\}$ and $f_{i,\alpha_3} = \max_{j \in \Omega_2 \cap \Omega_3} f_{i,j} \leq 0$. Note that $\alpha_2 \in \Omega_2 \subseteq \Omega$. From Condition A, we know that $\Omega_2 \cap \Omega_3 \geq \tau(n-1)/2$ and $\Omega_2^c \cap \Omega_3 \leq n_{\alpha_2} - \tau(n-1)/2 \leq (\rho_{\max} - \tau/2)(n-1)$. Thus,

$$-(f_{i,\alpha_2} + f_{i,\alpha_3}) \frac{m|\Omega_2 \cap \Omega_3|}{Mn_{\alpha_2}} - (f_{i,\alpha_2} + f_{i,\alpha}) \frac{M|\Omega_2^c \cap \Omega_3|}{mn_{\alpha_2}} \leq b,$$

or equivalently,

$$-(f_{i,\alpha_2} + f_{i,\alpha_3}) \frac{m|\Omega_2 \cap \Omega_3|}{Mn_{\alpha_2}} \leq b + (f_{i,\alpha_2} + f_{i,\alpha}) \frac{M|\Omega_2^c \cap \Omega_3|}{mn_{\alpha_2}}.$$

Hence

$$-(f_{i,\alpha_2} + f_{i,\alpha_3}) \leq \frac{2Mb\rho_{\max}}{m\tau} + \frac{4M^2\rho_{\max}(\rho_{\max} - \tau/2)b}{m^2\tau^2}.$$

By (19), we have

$$\begin{aligned} f_{i,\alpha} &\leq -f_{i,\alpha_2} + \frac{2\rho_{\max}Mb}{m\tau} \\ &\leq -(f_{i,\alpha_2} + f_{i,\alpha_3}) + \frac{2\rho_{\max}Mb}{m\tau} \\ &\leq \frac{4M^2\rho_{\max}(\rho_{\max} - \tau/2)b}{m^2\tau^2} + \frac{4\rho_{\max}Mb}{m\tau}. \end{aligned}$$

This completes the proof.

3. Discussion

The matrix S which we use to approximate the inverse of the matrix T takes the form of $I + H_c$, where each element of H_c is c . If $c > 0$, then S is a class of preconditioners for M -matrices [18]. In our situation, $c < 0$ since S is a matrix with non-negative elements. The bound on the approximation error in Theorem 1 depends on m , M and n . When m and M are bounded by a constant, all the elements of $T^{-1} - S$ are of order $O(1/(n-1)^2)$ as $n \rightarrow \infty$, uniformly. Therefore we conjecture that T may be inverse M -matrices. The interested readers can refer to [2, 13].

When proving the asymptotic normality of the maximum likelihood estimate in the β -model with a diverging dimension n , Yan, Xu and Yang [16] need to consider the approximation inverse of a Fisher information matrix satisfying the conditions (1) due to that it doesn't have an explicit expression but some special cases, and use S as its workable substitute. Theorem 2 is used to a situation with sparse statistical experiments for the β -model in our future work. Moreover, in a work on strict diagonally dominant M -matrices with certain bounded conditions, Simons and Yao [14] derived their good approximated inverses and proved the approximated errors are order $1/n^2$ as well, which is crucial in proving that the maximum likelihood estimate in the Bradley-Terry model is asymptotical normality with a large dimension [15].

Finally, we illustrate by an example that the bound on the approximation error in Theorem 2.1 is optimal in the sense that any bound in the form of $C(m, M)/f(n)$ requires $f(n) = O((n-1)^2)$ as $n \rightarrow \infty$. Assume that the matrix T consists of the elements: $t_{1,1} = (n-1)M$; $t_{1,j} = t_{j,1} = M$, $j = 2, \dots, n$ and $t_{i,i} = (n-1)m$, $i = 2, \dots, n$; $t_{i,j} = m$, $i, j = 2, \dots, n$; $i \neq j$, which satisfies (1). By the Sherman-Morrison formula, we have

$$\begin{aligned} (T^{-1})_{1,1} &= \frac{1}{(n-2)M} - \frac{1}{2(n-1)(n-2)M}, \\ (T^{-1})_{i,1} &= -\frac{1}{2(n-1)(n-2)M}, \quad i = 2, \dots, n, \end{aligned}$$

and for $i \geq j \geq 2$,

$$(T^{-1})_{i,j} = \frac{\delta_{i,j}}{(n-2)m} - \frac{1}{2(n-1)(n-2)m}.$$

In this case, the elements of S are

$$\begin{aligned} S_{1,1} &= \frac{\delta_{1,1}}{(n-1)M} - \frac{1}{(n-1)(M+(n-1)m)} \\ S_{i,i} &= \frac{\delta_{i,i}}{(n-1)m} - \frac{1}{(n-1)(M+(n-1)m)}, i = 2, \dots, n, \\ S_{i,j} &= \frac{\delta_{i,j}}{(n-1)m} - \frac{1}{(n-1)(M+(n-1)m)}, \quad i, j = 1, \dots, n; i \neq j. \end{aligned}$$

It is easy to show that the bound of $\|T^{-1} - S\|$ is $O(\frac{1}{(n-1)^2 m^2})$. This suggests that the rate $1/(n-1)^2$ is optimal. On the other hand, there is a gap between $1/m^2$ and $C(m, M) = O(M^2/m^3)$ which implies that there might be space for improvement. It is interesting to see if the bounds in Theorem 2.1 can be further relaxed as well as in Theorem 2.2.

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